STAT 620: Asymptotic Statistics **Spring 2022**

Lecture: Jan 18

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1 Different notions of convergence for a sequence of random variables

A sequence of random variables $\{X_n\}$ is said to converge to X

(i) in probability (i.p.) if $\forall \epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0.$$

This is denoted by $X_n \overset{p}{\to} X$.

(ii) almost surely (a.s.) if

$$P\Big(\lim_{n\to\infty} X_n = X\Big) = 1.$$

This is denoted by $X_n \overset{a.s.}{\to} X$.

(iii) in L_p if

$$\|X_n - X\|_p = \Big[\mathbb{E}|X_n - X|^p\Big]^{1/p} \to 0.$$

This is denoted by $X_n \overset{L_p}{\to} X$.

(iv) in distribution if

$$F_n(x) = P(X_n \leq x) \to F(x) = P(X \leq x),$$

for all $x \in \mathcal{C}(F)$, where $\mathcal{C}(F)$ is the set of all continuity points of F. This is denoted by $X_n \overset{d}{\to} X$. In this case, X_n and X need not be defined on the same probability space. Each X_n may be in a different probability space.

2 Cramer-Wald device

Suppose $\{X_n\}$ is a sequence of k dimensional random vectors and X is a k dimensional random vectors. Then $X_n \overset{d}{\to} X$ iff $\alpha^\top X_n \overset{d}{\to} \alpha^\top X$ for all $\alpha \in \mathbb{R}^k$.

3 Continuous mapping theorem

Let X_n, X be random variables defined on a metric space S. Suppose $g : S \to S'$ has the set of discontinuous points D_g such that $P(X \in D_g) = 0$. Then

$$\begin{aligned} X_n \to^d X &\quad\Rightarrow\quad g(X_n) \to^d g(X); \\ X_n \to^p X &\quad\Rightarrow\quad g(X_n) \to^p g(X); \\ X_n \to^{a.s} X &\quad\Rightarrow\quad g(X_n) \to^{a.s} g(X). \end{aligned}$$

4 Slutsky's theorem

Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables. If $\{X_n\}$ converges in distribution to a random variable X and $\{Y_n\}$ converges in probability to a constant c, then

$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} \xrightarrow{d} \begin{bmatrix} X \\ c \end{bmatrix}$.

Applying continuous mapping theorem, we also have

- $X_n + Y_n \xrightarrow{d} X + c$
- $X_n Y_n \xrightarrow{d} cX$
- $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$ if $c \neq 0$

5 Uniform integrability

A sequence of random variables $\{X_n\}$ is defined to be uniform integrable (u.i.) if

$\sup_n \mathbb{E}|X_n| \mathbf{1}\{|X_n| > M\} \to 0$ as $M \to \infty$.

Recall from STAT 614 that $X_n \xrightarrow{p} X$ and X_n is u.i. together implies that $X_n \xrightarrow{L_1} X$.

6 Law of Large Numbers

Suppose, $X_1, X_2, \dots, X_n$ are i.i.d random vectors with $\mathbb{E}X_1 = \mu$. Then

- The **Weak Law of Large Numbers** states that

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} \mu.$$

- The **Strong Law of Large Numbers** states that

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mu.$$

7 Central Limit Theorem

Let X_i be a sequence of independent random variables defined on the same probability space. Suppose $\mu_i = E[X_i]$ and $\sigma_i^2 = \text{var}(X_i)$ exist and let $s_n^2 = \sum_{i=1}^n \sigma_i^2$. If Lindeberg's condition holds:

$$\frac{1}{s_n^2} \sum_{i=1}^{n} E[(X_i - \mu_i)^2 \mathbf{1}\{|X_i - \mu_i| > \epsilon s_n\}] \to 0$$

for all $\epsilon > 0$, then we have

$$\frac{\sum_{i=1}^n (X_i - \mu_i)}{s_n} \to^d N(0, 1).$$

8 Portmanteau Theorem

For any random vectors $X_1, X_2, \ldots, X_n$ and X, the following statements are equivalent:

1. $P(X_n \leq x) \to P(X \leq x)$ for all continuous points x of $P(X \leq x)$.
2. $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for all bounded and continuous functions f.
3. $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for all bounded and Lipschitz functions f.
4. $\liminf_n P(X_n \in G) \geq P(X \in G)$ for all open set G.
5. $\limsup_n P(X_n \in F) \leq P(X \in F)$ for all closed set F.
6. $\lim_n P(X_n \in B) = P(X \in B)$ for all Borel sets B with $P(X \in \partial B) = 0$, where ∂B denotes the boundary of the set B.